Invariant-based reconstruction of bidimensionnal elasticity tensors

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Abstract

In the present paper a method to reconstruct the 2D elasticity tensor from observations is proposed. The novelty of the method, compared to other ones, is that this approach is based on the identification of the invariants, rather than the components, of the elasticity tensor. The main advantage is that all the information concerning the material, such as its symmetry class and the orientation of the tested sample, are obtained during the reconstruction process. We believe that such an approach based on intrinsic quantities may find interesting applications for the identification of mechanical parameters based on full-field measurements.

Keywords: Symmetry classes, Invariants, Elasticity

1. Introduction

These last years have seen a renew of interest toward the use of tensor invariants in continuum mechanics [13, 12, 4, 29, 17]. This renew is driven by the will to better understand links that exist between microstructure and overall elastic behavior. Since, and contrary to tensor components, tensor invariants provide intrinsic information on the material, their use is more appealing for applications in which material evolution occurs: damage mechanics [7, 18], optimal conception [26].

The topic of the present article is slightly different from the previously evoked applications and concerns the labelling of elastic materials from the invariants of the elasticity tensors. This subject was, up to authors knowledge, first discussed by Boehler et al. [10] in the context of 3D elasticity. And if, despite some progresses [10, 4, 28], the problem remains open in 3D space[1], the situation is clear for 2D elasticity. In this case the set of fundamental polynomial invariants (also referred to as an integrity basis) is known since the end of the 90' [20, 8, 33]. Those invariants

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[1] In fact the question has been theoretically solved in [28] but some works are still needed to put the result into an amenable form.

were discovered and rediscovered at different times, hence multiple names are attached to them. But as demonstrated in an important review paper [17] these different approaches are strictly equivalent.

The aim of the present paper is two-fold. First, to give a *mechanical interpretation* of fundamental polynomial invariants. Such a picture is important in order to promote their use. Second, to propose *numerical experiments* to directly measure them. This point proves that, at least in 2D, such quantities are observable, and may give some hints on how to design experimental testing devices to quantify them. Eventually, we show how to reconstruct elasticity tensors from invariants. Since symmetry classes are encoded by relations between invariants, within this approach all the information about the elastic material is directly obtained. We believe that this method may find interesting applications in the development of identification methods of mechanical parameters based on full-field measurements [5].

The paper is organized as follows. In the first section basic facts about the 2D elasticity tensor and its symmetry classes are summed-up. The next section introduces the harmonic decomposition of the tensor and gives the expressions of polynomial invariants. The link between invariants and symmetry classes is presented. In §.4 the harmonic decomposition is analysed in terms of mechanical quantities such as the stress-tensor, and the strain-energy. As a result the physical meaning of the elementary invariants is clarified and numerical experiments to measure them (§.5) are proposed. In §.6, it is shown how to reconstruct the elasticity tensor from these data. Some concluding remarks close this paper. Appendices are devoted to detail some technical aspects of the present study.

Notations: The following matrix groups will be considered:

- $\mathrm{GL}(2)$: the group of invertible transformations of $\mathbb{R}^2$, i.e. if $\mathrm{F} \in \mathrm{GL}(2)$ then $\det(\mathrm{F}) \neq 0$;
- $\mathrm{O}(2)$: the orthogonal group, that is the group of all isometries of $\mathbb{R}^2$ i.e. $\mathrm{Q} \in \mathrm{O}(2)$ iff $\det(\mathrm{Q}) = \pm 1$ and $\mathrm{Q}^{-1} = \mathrm{Q}^T$, where the superscript ^{T} denotes the transposition. As a matrix group $\mathrm{O}(2)$ can be generated by:

$$\underset{\sim}{\mathrm{r}_\theta} := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \ 0 \leq \theta < 2\pi, \quad \text{and} \quad \underset{\sim}{\sigma_{\mathrm{x}}} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  in which $\underset{\sim}{\mathrm{r}_\theta}$ is a rotation of θ angle and $\underset{\sim}{\sigma_{\mathrm{x}}}$ is the reflection across the x axis;
- $\mathrm{SO}(2)$: the special orthogonal group, i.e. the subgroup of $\mathrm{O}(2)$ of elements satisfying $\det(\mathrm{Q}) = 1$. This is the group of 2D rotations generated by $\underset{\sim}{\mathrm{r}_\theta}$;

In this work, zero-th, first, second, fourth and eighth order tensors are denoted by a, $\underline{a}$, $\underset{\sim}{a}$, $\underset{\approx}{a}$ and $\underset{\frown}{a}$ respectively. The simple, double and fourth contractions are written $\cdot$, : and :: accordingly. In index form with respect to an orthonormal Cartesian basis, these notations correspond to

$$\underline{a} \cdot \underline{b} := a_i b_i, \quad \underset{\sim}{a} : \underset{\sim}{b} := a_{ij} b_{ij}, \quad \underset{\approx}{a} :: \underset{\approx}{b} := a_{ijkl} b_{ijkl}$$

where repeated indices are summed up. The sign := defines the quantity on the left-hand side. The tensor product is classically denoted $\otimes$ while $\otimes^n$ stands for its n-th power. Vector spaces will be denoted using blackboard bold fonts, and their tensorial order indicated by using formal indices. When needed index symmetries are expressed as follows: (..) indicates invariance under permutation of the indices in parentheses, and $\underline{..}$ indicates invariance with respect to permutations of the underlined blocks.

2. The space of 2D elasticity tensors

In the following $\mathcal{E}^2$ will be the physical 2-dimensional Euclidean space. The anisotropic elastic behavior will be introduced within this framework. The definitions of material and physical symmetries will be recalled, and the list of bi-dimensional elastic symmetry classes finally provided.

2.1. Constitutive law

In the theory of linear elasticity for an anisotropic homogeneous body, the constitutive law is a local linear relation between the second-order symmetric Cauchy stress tensor $\underset{\sim}{\sigma}$ and the second-order symmetric infinitesimal strain tensor $\underset{\sim}{\varepsilon}$:

$$\sigma_{ij} := C_{ijlm} \varepsilon_{lm} \tag{1}$$

Since both $\underset{\sim}{\sigma}$ and $\underset{\sim}{\varepsilon}$ are symmetric with respect to index permutations, the elasticity tensor inherits these *minors* symmetries:

$$C_{ijlm} = C_{jilm} = C_{jiml}$$

These symmetries are condensed in the following notation: $C_{(ij)(kl)}$. Due to the potential energy associated to the elastic behavior another index symmetry has to be taken into account:

$$C_{ijlm} = C_{lmij}$$

This so-called *major* symmetry is encoded in the notation: $C_{\underline{ij}\ \underline{kl}}$. Hence, combined with the minor ones, we obtain the elastic index symmetries: $C_{\underline{(ij)}\ \underline{(kl)}}$. As a consequence the vector space of elasticity tensors is defined as[2]

$$\mathbb{E}\mathrm{la} := \{\underset{\approx}{\mathrm{C}} \in \otimes^4\mathbb{R}^2 | C_{\underline{(ij)}\ \underline{(kl)}}\}, \quad \dim \mathbb{E}\mathrm{la} = 6$$

This formulation of the elasticity law (1) is valid for any anisotropic material. In the following subsection the concept of material and physical symmetries will be recapped.

2.2. Material symmetry & Physical symmetry

Let consider a body as a compact subset $\mathcal{D}_0$ of $\mathcal{E}^2$ having a microstructure $\mathcal{M}$ attached to any of its material points $P \in \mathcal{D}_0$. Those points are located with respect to a reference frame $(\mathcal{R})$. The microstructure, which is represented by an open subset of $\mathbb{R}^2$ over P, describes the local organisation of the matter at scales below the one used for the continuous description (see fig.1):

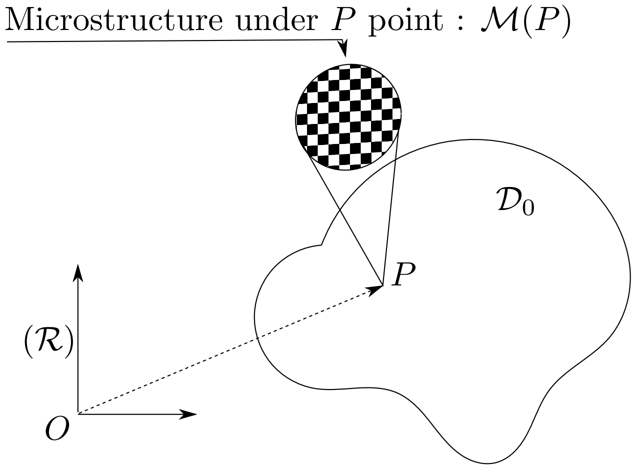


Figure 1: What is hidden below a material point

For crystalline materials the microstructure is the crystal lattice, for polymers the organisation of polymeric chains, ... As for crystals, microstructures can possess invariance properties with respect to orthogonal transformations $\mathrm{Q} \in \mathrm{O}(2)$. Hence at each material point P, the set of such transformations forms a point group $\mathrm{G}_{\mathcal{M}(P)} \subseteq \mathrm{O}(2)$ which describes the local material symmetries, formally

$$\mathrm{G}_{\mathcal{M}(P)} := \{Q \in \mathrm{O}(2), \quad Q \cdot \mathcal{M}(P) = \mathcal{M}(P)\}$$

At the continuous macroscopic scale the detailed description of the microstructure is lost, and information on the microstructure is contained in $\mathrm{G}_{\mathcal{M}}(P)$. In the case of an homogeneous medium the point dependence vanishes and $\mathrm{G}_{\mathcal{M}(P)} = \mathrm{G}_{\mathcal{M}}$.

[2]It is worth noting that for being admissible, an elasticity tensor should further be positive definite. Since this point plays no role in the present discussion this restriction will not be considered here.

Linear constitutive laws are encoded by tensors: a tensor T is attached to each material point P of $\mathcal{D}_0$, resulting in a tensor field that need not be continuous in general. In the present situation, that is for the linear elasticity, the heterogeneous behavior is described by a field of fourth-order tensor $\underset{\approx}{\mathrm{C}}(P)$. If the material is homogeneous this tensor field is constant, and a "unique" $\underset{\approx}{\mathrm{C}}$ describes the behavior. This hypothesis of material homogeneity will be assumed for the rest of the paper.

As the material element is transformed, the physical property described over it is also transformed in some related way. Hence a notion of physical symmetry group has to be introduced. Since elasticity is described by a fourth-order tensor, the physical symmetry group is the symmetry group of this tensor, $\mathrm{G}_{\underset{\approx}{\mathrm{C}}} \subseteq \mathrm{O}(2)$:

$$\mathrm{G}_{\underset{\approx}{\mathrm{C}}} := \{Q \in \mathrm{O}(2)\,|Q_{io}Q_{jp}Q_{kq}Q_{lr}C_{opqr} = C_{ijkl}\,\}. \tag{2}$$

The link between these two notions is given by the Curie principle which states that the material symmetry group (cause) is included in the physical symmetry group (consequence):

$$\mathrm{G}_{\mathcal{M}} \subseteq \mathrm{G}_{\underset{\approx}{\mathrm{C}}}$$

More details concerning those notions can be be found in Zheng and Boehler [38].

2.3. Symmetry classes and strata

The notion of symmetry group, as defined in the previous subsection, is relative to a specific orientation of the material with respect to a given reference frame. If the material is rotated both the elasticity tensor and its symmetry group will be transformed. Since the nature of the material is left unchanged by this transformation those objects are not appropriate to characterize intrinsically elastic materials. The collection of all elasticity tensors obtained from $\underset{\approx}{\mathrm{C}}$ by O(2)-operations constitutes its O(2)-orbit:

$$\mathrm{O}_{\underset{\approx}{\mathrm{C}}} := \{C^{\star}_{ijkl} \in \mathbb{E}\mathrm{la}|\exists Q \in \mathrm{O}(2), C^{\star}_{ijkl} = Q_{io}Q_{jp}Q_{kq}Q_{lr}C_{opqr}\}$$

The $\underset{\approx}{\mathrm{C}}$-orbit represents all the elasticity tensors associated to $\underset{\approx}{\mathrm{C}}$, and hence is O(2)-invariant. As a consequence $\mathrm{O}_{\underset{\approx}{\mathrm{C}}}$ characterizes the elastic material which was represented by $\underset{\approx}{\mathrm{C}}$ in a specific orientation w.r.t $(\mathcal{R})$. In the same way two symmetry groups $\mathrm{G}_{\underset{\approx}{\mathrm{D}}}$, $\mathrm{G}_{\underset{\approx}{\mathrm{C}}}$ are said O(2)-conjugate, if

$$\exists Q \in \mathrm{O}(2),\ \mathrm{G}_{\underset{\approx}{\mathrm{D}}} = Q\mathrm{G}_{\underset{\approx}{\mathrm{C}}}Q^T. \tag{3}$$

The symmetry class of $\underset{\approx}{\mathrm{C}}$ is the set $[\mathrm{G}_{\underset{\approx}{\mathrm{C}}}]$ of O(2)-subgroups conjugate to $\mathrm{G}_{\underset{\approx}{\mathrm{C}}}$:

$$[\mathrm{G}_{\underset{\approx}{\mathrm{C}}}] := \{\mathrm{G} \subseteq \mathrm{O}(2)\,\Big|\,\mathrm{G} = Q\mathrm{G}_{\underset{\approx}{\mathrm{C}}}Q^T, Q \in \mathrm{O}(2)\,\}. \tag{4}$$

In other words, the symmetry class of $\underset{\approx}{\mathrm{C}}$ is its symmetry group modulo its orientation. If two tensors belong to the same orbit, they have the same symmetry class, the converse is false.

In a bidimensional space, the symmetry class of a tensor is conjugate to a closed subgroup of O(2). The collection of these subgroups are known and are elements of the following set [1]:

$$\{\mathrm{Id}, \mathrm{Z}_2^{\sigma_x}, \mathrm{Z}_k, \mathrm{D}_k, \mathrm{SO}(2), \mathrm{O}(2)\}_{k \in \mathbb{N}_{>1}}$$

in which the following groups are involved:

- Id, the identity group;
- Z_k, the cyclic group[3] with k elements generated by $r_{2\pi/k}$, a rotation angle of $2\pi/k$;
- SO(2), the infinitesimal rotation group, the cyclic limit group for $k \to \infty$;
- $\mathrm{Z}_2^{\sigma_x}$, where σ_x denotes the mirror transformation through the x axis;
- D_k, the dihedral group with $2k$ elements generated by $r_{2\pi/k}$ and σ_x;
- O(2), the infinitesimal orthogonal group, the dihedral limit group for $k \to \infty$.

It has been demonstrated that in 2D, there is only 4 different possibilities for the symmetry class of an elasticity tensor [21, 33, 17]:

Name	Digonal	Orthotropic	Tetragonal	Isotropic
$[\mathrm{G}_{\underset{\approx}{\mathrm{C}}}]$	$[\mathrm{Z}_2]$	$[\mathrm{D}_2]$	$[\mathrm{D}_4]$	$[\mathrm{O}(2)]$
$\#_{\mathrm{indep}}(\underset{\approx}{\mathrm{C}})$	6 (5)	4	3	2

According to that classification, $\mathbb{E}\mathrm{la}$ can be divided into 4 sets that regroup tensors of each type. More formally, $\Sigma_{[G]}$ is the stratum[4] of tensors having symmetry group conjugate to $G \subseteq \mathrm{O}(2)$. In other terms:

$$\mathbb{E}\mathrm{la} = \Sigma_{[\mathrm{Z}_2]} \cup \Sigma_{[\mathrm{D}_2]} \cup \Sigma_{[\mathrm{D}_4]} \cup \Sigma_{[\mathrm{O}(2)]}$$

which means that any 2D elastic tensor belong uniquely to one of these strata. Tensor invariants that will be introduced in the next section allow to label elastic materials. Since those quantities are unchanged if the material is rotated/flipped, they provide an interesting way to designate elastic materials. Furthermore certain sets of invariants, referred to as functional basis, have the property to *separate orbits*, i.e. to uniquely label elastic materials.

[3] It has to be noted that Z_2^{π} and Z_2 are isomorphic as group but not conjugate.

[4] It is worth noting that strata are not linear subspaces of $\mathbb{E}\mathrm{la}$. A detail discussion on the geometry of strata is provided in Auffray et al. [4].

3. Fundamental invariants

When speaking of tensor invariants some points have to be specified:

Group action Since the notion of an *invariant* quantity is relative to a group of transformations, the group has to be indicated. Invariants under O(2)− or SO(2)−action, for example, are not the same [33, 17];

Functional nature The functional form under which invariants are sought need to be indicated. According to the type of functions the size of the generating basis may vary widely [34, 9]. Bases for polynomial invariants are usually referred to as *integrity* bases, while those for general functions are called *functional*[5] [35].

It is known that an integrity basis is a functional one, but the converse is false. Interest in functional basis are related to their property to separate orbits [35]. Any set that separates orbits can be used to label elastic materials in an unique fashion. This is the core point of our interest in invariant theory.

Albeit being essential those points are, nevertheless, not always clearly specified in mechanical publications. In the present paper, attention will be drawn to polynomial invariants under SO(2)- and O(2)-action. A well-known set of invariants is constituted of the Kelvin moduli associated with the eigenvalues of the elasticity tensor. It is important to note that those quantities are algebraical but non polynomial. Furthermore the group action is no more O(2) in this case, but O(3) [37]. Hence our object is not the study of Kelvin invariants even if relations exist between these two sets.

To obtain polynomial invariants, the elasticity tensor has first to be decomposed into elementary tensors irreducible under considered group action. This first step is generally referred to as the harmonic decomposition [25, 14, 33]. Polynomial invariants are then constructed from the elements of that decomposition. Remarks concerning non-polynomial invariants that can be found in the literature [13] are made. Following a geometric interpretation proposed by Forte and Vianello [15, 17] a *vector* representation of the harmonic decomposition is introduced.

[5] It should be noted that for functional bases, the size of bases vary according to the functional nature of the basis elements. But, in practice those basis elements are nearly always taken polynomial [36].

3.1. Irreducible decomposition

To determine its integrity basis, the elasticity tensor need to be decomposed into irreducible components[6]. This is achieved through a two-step process [33, 23, 17, 24]:

1. The tensor is decomposed into sub-tensors having same elementary index symmetries;
2. Traces are removed from the obtained set of intermediate tensors.

This process is summed-up in the following diagram:

$$\begin{tikzcd}
 & \mathbb{E}\mathrm{la} \arrow[dl, "\mathrm{Sym}"'] \arrow[dr, "\mathrm{Asym}=\mathrm{Id}-\mathrm{Sym}"] & \\
\mathbb{S}_{(ijkl)} \arrow[d, "\mathcal{H}"] & & \mathbb{R}_{ijkl} \arrow[d, "\mathcal{H}"] \\
\mathbb{K}^4 \oplus \mathbb{K}^2 \oplus \mathbb{K}^0 & & \mathbb{K}^0
\end{tikzcd}$$

in which Sym and Id denote the full-symmetrizing and identity operations, while $\mathcal{H}$ stands for the removing of successive traces. Spaces that appear in this decomposition are:

- $\mathbb{S}_{(ijkl)}$ and $\mathbb{R}_{ijkl}$: those spaces are GL(2)-invariants [23, 24]. In reference to an historical dispute concerning the structure of the elasticity tensor[7], the complete symmetric part ($S_{ijkl} \in \mathbb{S}_{(ijkl)}$) will be referred to as the Cauchy part, while the remainder ($R_{ijkl} \in \mathbb{R}_{ijkl}$) will be called the non-Cauchy part.

- $\mathbb{K}^k$: those spaces are O(2)-invariant and referred to as harmonic tensor spaces. The main properties of harmonic tensors are of being totally symmetric and traceless. Their dimensions are[8]:

$$\dim \mathbb{K}^k = \begin{cases} 2, & k \geq 1 \\ 1, & k = 0, -1 \end{cases} \tag{5}$$

More explicitly, the tensor elasticity tensor is first decomposed into its Cauchy and non-Cauchy parts:

$$C_{\underline{(ij)}\ \underline{(kl)}} = S_{(ijkl)} + R_{\underline{(ij)}\ \underline{(kl)}}$$

[6] By irreducible components we mean sub-tensors that transform in elementary way under the considered group action. For a deeper introduction we refer the reader to the following references [31, 14].

[7] In short, this quarrel opposed the French school of mechanics (Navier, Cauchy) to the English one (Green) on the number of elastic constants needed to properly set an elastic problem. For a historical discussion on that interesting topics we refer to [30, 11].

[8] The uni-dimensional space $\mathbb{K}^{-1}$ contains pseudo-scalars, i.e. quantities that change sign if the space orientation is reversed. Spaces of that type are not involved in the harmonic structure of the elasticity tensor.

in which:

$$S_{(ijkl)} = \frac{1}{3}\left(C_{ijkl} + C_{iklj} + C_{iljk}\right) \quad ; \quad R_{\underline{(ij)}\ \underline{(kl)}} = \frac{1}{3}\left(2C_{ijkl} - C_{iklj} - C_{iljk}\right)$$

Then all traces are removed to obtain harmonic tensors [33]:

$$\begin{aligned}
D_{ijkl} &= C_{ijkl} - \frac{1}{6}\left(\delta_{ij}C_{kplp} + \delta_{kl}C_{ipjp} + \delta_{ik}C_{lpjp} + \delta_{lj}C_{ipkp} + \delta_{il}C_{jpkp} + \delta_{jk}C_{iplp}\right) \\
&\quad + \frac{C_{pqpq}}{12}\left(5\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right) - \frac{C_{ppqq}}{8}\left(3\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right) \\
a_{ij} &= \frac{1}{12}\left(2C_{ipjp} - C_{pqpq}\delta_{ij}\right) \\
\lambda &= \frac{1}{8}\left(3C_{ppqq} - 2C_{pqpq}\right) \\
\mu &= \frac{1}{8}\left(2C_{pqpq} - C_{ppqq}\right)
\end{aligned}$$

Doing some algebra the following expression is reached[9]

$$C_{ijkl} = D_{ijkl} + \frac{1}{6}\left(\delta_{ij}a_{kl} + \delta_{kl}a_{ij} + \delta_{ik}a_{jl} + \delta_{jl}a_{ik} + \delta_{il}a_{jk} + \delta_{jk}a_{il}\right) + \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad (6)$$

which can be compressed using the following notation: $\underset{\approx}{\mathrm{C}} = \phi(\underset{\approx}{\mathrm{D}}, \underset{\sim}{\mathrm{a}}, \lambda, \mu)$. The formula (6) is an isomorphism between the space of elasticity tensors and a direct sum of harmonic spaces:

$$\mathbb{E}\mathrm{la} \underset{\phi}{\simeq} \mathbb{K}^4 \oplus \mathbb{K}^2 \oplus 2\mathbb{K}^0$$

with the following essential property

$$\underset{\frown}{\mathrm{Q}}^{(4)} :: \underset{\approx}{\mathrm{C}} = \phi\left(\underset{\frown}{\mathrm{Q}}^{(4)} :: \underset{\approx}{\mathrm{D}}, \underset{\sim}{\mathrm{Q}}^{(2)} : \underset{\sim}{\mathrm{a}}, \lambda, \mu\right)$$

where

$$\mathrm{Q}^{(4)}_{ijklmnop} = Q_{im}Q_{jn}Q_{ko}Q_{lp} \quad ; \quad \mathrm{Q}^{(2)}_{ijmn} = Q_{im}Q_{jn}$$

The isomorphism ϕ that realizes the harmonic decomposition is not uniquely defined. If $\underset{\approx}{\mathrm{D}} \in \mathbb{K}^4$ and $\underset{\sim}{\mathrm{a}} \in \mathbb{K}^2$ are uniquely defined, the choice of the two isotropic parts is somehow arbitrary. Such a fact is indeed well-known since there is a lot of possible couples of isotropic parameters. For our needs, in the following, the bulk and shear moduli (K, G):

$$K := \lambda + \mu \quad ; \quad G := \mu$$

will be preferred to the Lamé moduli.

[9]It is worth noting that the structure of the harmonic decomposition depends on the dimension of the physical space. Hence the harmonic decomposition of 3D elasticity tensors is slightly different [6, 14].

3.2. Integrity basis

Integrity bases for SO(2)- and O(2)-action on the space of plane elasticity tensors are known since the second-half of the 90' [8, 33]. While expressed in slightly different manners in cited publications, these bases are constituted by the following invariants:

- 4 simple invariants:

$$I_1 = K \quad ; \quad J_1 = G \quad ; \quad I_2 = a_{pq}a_{pq} \quad ; \quad J_2 = D_{pqrs}D_{pqrs}$$

- 2 joint invariants:

$$I_3 = a_{pq}D_{pqrs}a_{rs} \quad ; \quad J_3 = R_{pq}a_{qr}D_{prst}a_{st}$$

  with

$$R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

It can be noted that those invariants are related through the following polynomial relation (sygyzy):

$$2I_3^2 + 2J_3^2 - I_2^2 J_2 = 0$$

The sets:

- $(I_1, J_1, I_2, J_2, I_3)$ is an integrity basis for O(2)-action;
- $(I_1, J_1, I_2, J_2, I_3, J_3)$ is an integrity basis for SO(2)-action;

The demonstration of these results can be found in [33]. Non-polynomial functions of the integrity basis elements still separate the orbits. The following set of non polynomial invariants can be considered

- 4 simple invariants:

$$i_1 = K \quad ; \quad j_1 = G \quad ; \quad i_2 = \sqrt{I_2} \quad ; \quad j_2 = \sqrt{J_2}$$

- 2 joint invariants:

$$i_3 = \frac{\sqrt{2}I_3}{I_2\sqrt{J_2}} \quad ; \quad j_3 = \frac{\sqrt{2}J_3}{I_2\sqrt{J_2}}$$

  Those invariants can naturally be understood as the cos and sin of an angle between "vectors" associated to $\underset{\sim}{a}$ and $\underset{\approx}{D}$. Hence, those invariants are related through the following relation:

$$i_3^2 + j_3^2 = 1$$

From this set, and using the *arcos* function, the non-polynomial invariants of Vannucci are retrieved [13]. It has to be noted that, contrary to polynomial invariants, those ones may not be globally defined and have local domains. This fact is clear from the definitions of i_3 and j_3.

3.3. Vector type interpretation

Following a geometric picture proposed by Forte and Vianello [33, 15, 17], in 2D harmonic tensors can be represented as vectors in appropriate orthonormal bases. This representation will be useful in section §.6 to identify the angular orientation of an elastic material with respect to an arbitrary frame of reference.

Following this idea, $\underset{\sim}{\mathrm{a}} \in \mathbb{K}^2$ can be associated to the following vector:

$$\underline{\tilde{\mathrm{a}}} = \begin{pmatrix} \sqrt{2} a_1 \\ \sqrt{2} a_2 \end{pmatrix}$$

in the basis $(\underline{\mathrm{E}}_1, \underline{\mathrm{E}}_2)$:

$$\underline{\mathrm{E}}_1 := \frac{\sqrt{2}}{2}\left(e_1 \otimes e_1 - e_2 \otimes e_2\right) \quad ; \quad \underline{\mathrm{E}}_2 := \frac{\sqrt{2}}{2}\left(e_1 \otimes e_2 + e_2 \otimes e_1\right)$$

The angle between $\underline{\tilde{\mathrm{a}}}$ and $\underline{\mathrm{E}}_1$ will be denoted α.

And in the same way, $\underset{\approx}{\mathrm{D}}$ can be represented by the following vector:

$$\underline{\tilde{\mathrm{D}}} = \begin{pmatrix} \sqrt{8} d_1 \\ \sqrt{8} d_2 \end{pmatrix}$$

in the basis $(\underline{\mathcal{E}_1}, \underline{\mathcal{E}_2})$

$$\begin{aligned} \underline{\mathcal{E}}_1 \quad := \quad & \frac{\sqrt{8}}{8}\left(e_1 \otimes e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 \otimes e_2 - e_1 \otimes e_1 \otimes e_2 \otimes e_2 - e_1 \otimes e_2 \otimes e_1 \otimes e_2\right. \\ & \left. - e_2 \otimes e_1 \otimes e_1 \otimes e_2 - e_2 \otimes e_1 \otimes e_2 \otimes e_1 - e_1 \otimes e_2 \otimes e_2 \otimes e_1 - e_2 \otimes e_2 \otimes e_1 \otimes e_1\right) \end{aligned}$$

$$\begin{aligned} \underline{\mathcal{E}}_2 \quad := \quad & \frac{\sqrt{8}}{8}\left(e_1 \otimes e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_1 \otimes e_2 \otimes e_1 + e_1 \otimes e_2 \otimes e_1 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 \otimes e_1\right. \\ & \left. - e_2 \otimes e_2 \otimes e_2 \otimes e_1 - e_2 \otimes e_2 \otimes e_1 \otimes e_2 - e_2 \otimes e_1 \otimes e_2 \otimes e_2 - e_1 \otimes e_2 \otimes e_2 \otimes e_2\right) \end{aligned}$$

The angle between $\underline{\tilde{\mathrm{D}}}$ and $\underline{\mathcal{E}}_1$ will be denoted β.

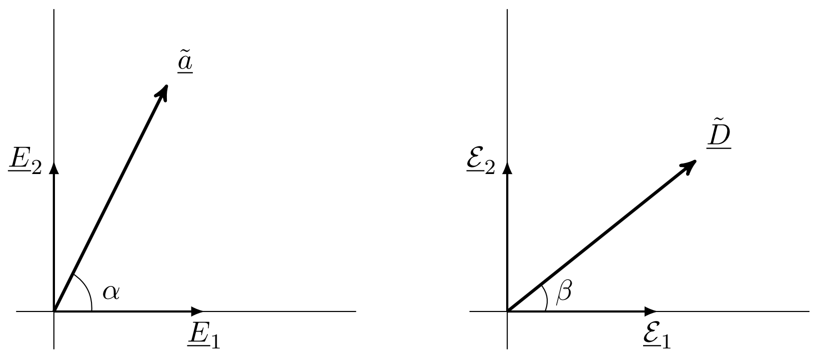


Figure 2: The vector representations of $\underline{\tilde{a}}$ and $\underline{\tilde{D}}$ in their respective orthogonal bases

3.4. Stratification

As discussed in the first section, the space of 2D elasticity tensors is divided into 4 strata:

$$\mathbb{E}\mathrm{la} = \Sigma_{[\mathrm{Z}_2]} \cup \Sigma_{[\mathrm{D}_2]} \cup \Sigma_{[\mathrm{D}_4]} \cup \Sigma_{[\mathrm{O}(2)]}$$

where notation $\Sigma_{[G]}$ indicates the set of tensors having its symmetry group conjugate to G. As indicated by the following graph, the vanishing of certain invariants indicates the symmetry class of the elasticity tensor:

$$\begin{tikzcd}
 & \Sigma_{[\mathrm{Z}_2]} \arrow[dl, "J_2=0"'] \arrow[dr, "J_3=0,\ J_2\neq 0"] & \\
\Sigma^d_{[\mathrm{D}_2]} \arrow[ddr, "I_2=0"] & & \Sigma^g_{[\mathrm{D}_2]} \arrow[d, "I_2=0"] \\
 & & \Sigma_{[\mathrm{D}_4]} \arrow[dl, "J_2=0"] \\
 & \Sigma_{[\mathrm{O}(2)]} &
\end{tikzcd}$$

In the former graph the sets $\Sigma^g_{[\mathrm{D}_2]}$ and $\Sigma^d_{[\mathrm{D}_2]}$ have been distinguished:

$\Sigma^g_{[\mathrm{D}_2]}$ Elements of this set are *generic*, or *non-degenerated*, orthotropic elasticity tensors. It only contains elements obtained just by imposing orthotropic invariance to generic anisotropic tensors. In such a situation the symmetry classes of $\underset{\approx}{\mathrm{D}}$ and $\underset{\sim}{\mathrm{a}}$ are, respectively, $[\mathrm{D}_4]$ and $[\mathrm{D}_2]$.

$\Sigma^d_{[\mathrm{D}_d]}$ Elements of this set are *degenerated* orthotropic tensors, they are not generic since extra restriction, i.e. other than invariance properties, are needed to define them. Those tensors correspond to what Vannucci called R_0-orthotropic tensors [32]. In this degenerated case the symmetry classes of $\underset{\approx}{\mathrm{D}}$ and $\underset{\sim}{\mathrm{a}}$ are, respectively, $[\mathrm{O}(2)]$ and $[\mathrm{D}_2]$.

It has to be pointed out that these elements belong to the same stratum:

$$\Sigma_{[\mathrm{D}_2]} = \Sigma^g_{[\mathrm{D}_2]} \cup \Sigma^d_{[\mathrm{D}_2]}$$

In 2D this is the unique example of *non-generic* tensor set, some prior computations reveal that this situation is much more frequent in 3D.

4. Mechanical content

Now that an integrity basis has been exhibited, and in order to set-up some experiments to measure them, it is important to have an insight into the mechanical content of its elements. The aim of the present section is to construct such a mechanical interpretation. In the following the elasticity tensor $\underset{\approx}{\mathrm{C}}$ will be observed through the second order stress tensor $\underset{\sim}{\sigma}$, and the scalar strain energy function W.

4.1. The Cauchy stress tensor

Let now express the Cauchy stress tensor in terms of irreducible decomposition of the elasticity tensor. In a first time σ_{ij} will be expressed as a whole before being split into deviatoric ($\sigma^d_{ij} \in \mathbb{K}^2$) and spherical ($\sigma^s_{ij} \in \mathbb{K}^0$) parts. Starting from the explicit harmonic decomposition (6), the stress tensor is obtained

$$\sigma_{ij} = D_{ijkl}\varepsilon_{kl} + \frac{1}{6}\left(a_{kl}\varepsilon_{kl}\delta_{ij} + a_{ij}\varepsilon_{pp} + 2(a_{ip}\varepsilon_{pj} + a_{jp}\varepsilon_{pi})\right) + \lambda\varepsilon_{pp}\delta_{ij} + 2\mu\varepsilon_{ij}$$

Let decompose the strain tensor into deviatoric (ε^d_{ij}) and spheric (ε^s_{ij}) components.

$$\varepsilon_{ij} = \varepsilon^d_{ij} + \varepsilon^s_{ij}$$

with

$$\varepsilon^d_{ij} = \varepsilon_{ij} - \frac{\varepsilon_{pp}}{2}\delta_{ij} \quad ; \quad \varepsilon^s_{ij} = \frac{\varepsilon_{pp}}{2}\delta_{ij}$$

thus

$$\sigma_{ij} = D_{ijkl}\varepsilon^d_{kl} + \frac{1}{6}\left(a_{kl}\varepsilon^d_{kl}\delta_{ij} + 2(a_{ip}\varepsilon^d_{pj} + a_{jp}\varepsilon^d_{pi})\right) + 2G\varepsilon^d_{ij} + \varepsilon_{pp}\left(\frac{1}{2}a_{ij} + K\delta_{ij}\right)$$

Hence the spheric and deviatoric stresses read [10]:

$$\begin{cases} \sigma^s_{ij} = \left(\frac{1}{2}a_{pq}\varepsilon^d_{pq} + K\varepsilon_{pp}\right)\delta_{ij} \\ \sigma^d_{ij} = D_{ijkl}\varepsilon^d_{kl} + 2G\varepsilon^d_{ij} + \frac{1}{2}\varepsilon_{pp}a_{ij} \end{cases}$$

From these expressions it can be observed that the tensor $\underset{\sim}{\mathrm{a}}$ generates shear stress from hydrostatic strain, and conversely hydrostatic stress from shear strain. Hence the invariant I_2, which is the Frobenius norm of $\underset{\sim}{\mathrm{a}}$, measures the level of transfer between hydrostatic and spheric modes.

4.2. The elastic energy

Using the harmonic decomposition the elastic energy reads:

$$2W = D_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \frac{1}{6}\left(a_{kl}\varepsilon_{kl}\varepsilon_{ii} + 2(a_{ip}\varepsilon_{pj} + a_{jp}\varepsilon_{pi})\varepsilon_{ij} + \varepsilon_{pp}a_{ij}\varepsilon_{ij}\right) + \lambda\varepsilon_{pp}\varepsilon_{qq} + 2\mu\varepsilon_{pq}\varepsilon_{pq}$$

[10] **In the computation the following term $\frac{1}{3}\left(a_{ip}\varepsilon^d_{pj} + a_{jp}\varepsilon^d_{pi} - a_{kl}\varepsilon^d_{kl}\delta_{ij}\right)$ appears in the expression of σ^d_{ij}. Some computations would show that this term is identically null**

and separating deviatoric and spheric contributions[11]:

$$2W = D_{ijkl}\varepsilon^d_{ij}\varepsilon^d_{kl} + 2G\varepsilon^d_{pq}\varepsilon^d_{pq} + a_{pq}\varepsilon^d_{pq}\varepsilon_{rr} + K\varepsilon_{pp}\varepsilon_{qq} \tag{7}$$

Using the harmonic decomposition of the elasticity tensor, the elastic energy naturally split into three contributions:

$$W = W^d + W^c + W^s$$

- The deviatoric energy[12]:

$$2W^d = D_{ijkl}\varepsilon^d_{ij}\varepsilon^d_{kl} + 2G\varepsilon^d_{pq}\varepsilon^d_{pq}$$

- The coupling energy:

$$2W^c = a_{pq}\varepsilon^d_{pq}\varepsilon_{rr}$$

- The spherical energy:

$$2W^s = K\varepsilon_{pp}\varepsilon_{qq}$$

As can be observed the spherical energy is purely isotropic and remains the same no matter how the material is anisotropic. At the opposite the coupling energy is purely anisotropic and, hence, vanishes for isotropic media. The deviatoric contribution, for its own, is defined by two irreducible components: one isotropic supplemented by an anisotropic part.
According to introduced energy partition (7) the mechanical meaning of most of those invariants are elucidated:

- J_1 measures the isotropic part of the deviatoricity of the material, while J_2 is the squared norm of its anisotropic part;
- I_1 measures the sphericity of the material;
- I_2 is the square of a norm that measures the amount of coupling energy in the material.

Last invariants I_3 and J_3 are less direct to interpret, their mechanical interpretations will be provided in the next section. As a consequence the different anisotropies can be classified according to the following scheme:

[11] In the computation a term proportional to $a_{ip}\varepsilon^d_{pj}\varepsilon^d_{ij}$ appears. This term is null in 2D, but not in higher dimension.

[12] To avoid any confusion, this does not imply that D_{ijkl} is positive definite, since D_{ijkl} is traceless it can not be positive definite.

	Deviatoric energy	Coupling	Spheric energy	Orientation
$\Sigma_{[\mathrm{Z}_2]}$	Anisotropic	Yes	Isotropic	Any
$\Sigma^g_{[\mathrm{D}_2]}$	Anisotropic	Yes	Isotropic	Aligned
$\Sigma^d_{[\mathrm{D}_2]}$	Isotropic	Yes	Isotropic	$\times$
$\Sigma_{[\mathrm{D}_4]}$	Anisotropic	No	Isotropic	$\times$
$\Sigma_{[\mathrm{O}(2)]}$	Isotropic	No	Isotropic	$\times$

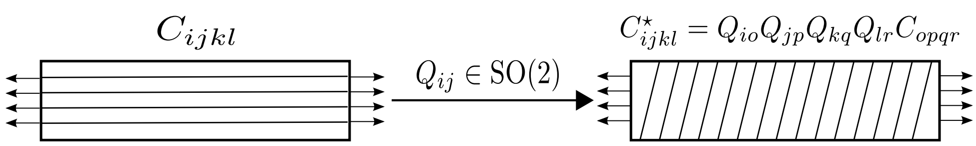
where $\times$ indicates that this information is meaningless. It can be observed that for the classes $[\mathrm{D}_4]$ and $[\mathrm{O}(2)]$ the coupling effect is null. Hence a characteristic of those classes is to produce stress tensors of the same type of the input strain tensors. Concerning elements in $\Sigma^d_{[\mathrm{D}_2]}$ they are somewhere in the middle between generic orthotropic elements ($\Sigma^g_{[\mathrm{D}_2]}$) and isotropic ones since they behave like isotropic elements with a spheric/deviatoric anisotropic coupling.

5. Reconstruction of the elasticity tensor

The aim of this section is to propose some numerical experiments that allow to, almost directly, measure the invariants of the elasticity tensor. Albeit being somewhat theoretical, this procedure is an interesting result since it is a constructive proof of the observability of the 2D elastic invariants. The numerical experiment we propose are optimal testing, it may give some hints for the designing of experimental procedure to measure those invariants.

Setting

Let us define a fixed reference frame equipped with a Cartesian basis, and consider an homogeneous anisotropic elastic material. No prior information is known about its micro-structure, and so on the privileged directions the material may possess. A first rectangular[13] sample is extracted out of the material and define a reference orientation for testing. The material is then rotated by an angle of θ and a second sample is extracted. These different samples are then tested with a fixed testing device. Hence between experiments only the orientation of the material within the sample varies.



Let introduce the following functions which express the stress tensor and the mechanical

[13]The shape of the sample has indeed no importance in the process.

energy as function of the angle θ[14]:

$$\begin{aligned}\underset{\sim}{\sigma}(\theta;\underset{\sim}{\varepsilon}) = \underset{\sim}{\sigma}(\theta;\varepsilon_{11},\varepsilon_{22},\varepsilon_{12}) &= \underset{\approx}{\mathrm{C}}(\theta) : \underset{\sim}{\varepsilon} \\ W(\theta;\underset{\sim}{\varepsilon}) = W(\theta;\varepsilon_{11},\varepsilon_{22},\varepsilon_{12}) &= \frac{1}{2}\underset{\sim}{\varepsilon} : \underset{\approx}{\mathrm{C}}(\theta) : \underset{\sim}{\varepsilon}\end{aligned}$$

with

$$\underset{\approx}{\mathrm{C}}(\theta) = \underset{\frown}{Q}^{(4)} :: \underset{\approx}{\mathrm{C}}$$

Experiments

To that aim let recap the shape of the elastic energy for a complete anisotropic material:

$$2W = D_{ijkl}\varepsilon^d_{ij}\varepsilon^d_{kl} + G\varepsilon^d_{pq}\varepsilon^d_{pq} + a_{pq}\varepsilon^d_{pq}\varepsilon_{rr} + K\varepsilon_{pp}\varepsilon_{qq}$$

The meaning of I_1 and J_1 are rather simple since we choose for them the well-known bulk and shear moduli. I_1 **is the strain-elastic energy associated with a unit equibiaxial strain-state:**

$$I_1 = K = \frac{1}{2}W(0;1,1,0)$$

To compute J_1 **we first need to impose a simple shear strain-state, in a second time the isotropic contribution to the elastic energy has to be singled out from the anisotropic contribution. To cancel the anisotropic part, the combination of the strain energy over two orientations of the sample has to be computed:**

$$J_1 = G = \frac{1}{4}\left(W(0;0,0,1) + W(\frac{\pi}{4};0,0,1)\right)$$

Since the other invariants measure anisotropic aspects, their computation involve the use of the stress tensor:

$$I_2 = \left\|\underset{\sim}{\sigma}^d(0;1,1,0)\right\|^2 \quad ; \quad J_2 = \frac{1}{2}\left(\left\|\underset{\sim}{\sigma}^d(0;0,0,1)\right\|^2 + \left\|\underset{\sim}{\sigma}^d\left(\frac{\pi}{4};0,0,1\right)\right\|^2\right) - 8J_1^2$$

in which $\|\cdot\|$ denotes the Frobenius norm. It can be observed that the quantity defined by $J_2 + 8J_1^2$ is a measure of the intensity of the deviatoricity of the material.

The joint invariant I_3 is computed in two times. First we determine the expression for a_{pq}:

$$\underset{\sim}{\mathrm{a}} = \underset{\sim}{\sigma}^d(0;1,1,0)$$

[14]The orientation $\theta = 0$ correspond to the material orientation of the reference sample, and do not have any intrinsic meaning.

which is then inserted into the energy expression

$$2W(0, a_1, -a_1, a_2) = \underset{\sim}{\mathrm{a}} : \underset{\sim}{\sigma}\,(0; a_1, -a_1, a_2) = D_{ijkl}a_{ij}a_{kl} + Ga_{pq}a_{pq}$$

where $\underset{\sim}{\mathrm{a}} = \begin{pmatrix} a_1 & a_2 \\ a_2 & -a_1 \end{pmatrix}$ and hence:

$$I_3 = 2W(0, \underset{\sim}{\mathrm{a}}) - J_1 I_2$$

This invariant can be interpreted by noting that $J_1 I_2$ correspond to the strain energy of an isotropic material loaded with $\underset{\sim}{\mathrm{a}}$

$$2W(0, \underset{\sim}{\mathrm{a}})^{\mathrm{O}(2)} = J_1 I_2$$

Hence

$$I_3 = 2\left(W(0, \underset{\sim}{\mathrm{a}}) - W(0, \underset{\sim}{\mathrm{a}})^{\mathrm{O}(2)}\right)$$

which gives a *physical* meaning to the last invariant. In the next section the role of this invariant will be illustrated on the reconstruction of elements in $\Sigma_{[\mathrm{D}_2]}$ (c.f. fig.3 and fig.4). For the last invariant J_3, the strategy is the same since we have

$$\underset{\sim}{\mathrm{a}}^{\star} : \underset{\sim}{\sigma}(0; \underset{\sim}{\mathrm{a}}) = R_{ip}a_{pj}D_{ijkl}a_{kl} + GR_{ip}a_{pj}a_{ij} = J_3$$

with $\underset{\sim}{\mathrm{a}}^{\star} = \underset{\sim}{\mathrm{R}} \cdot \underset{\sim}{\mathrm{a}}$, and the property $\underset{\sim}{\mathrm{a}}^{\star} : \underset{\sim}{\mathrm{a}} = 0$, which means that $\underset{\sim}{\mathrm{a}}^{\star}$ is orthogonal to $\underset{\sim}{\mathrm{a}}$. **From a practical point of view the strain-state associated to $\underset{\sim}{\mathrm{a}}$ might be very difficult to impose in practice to a sample.**

Strain-state control

The computation of invariants as proposed above supposes that a strain state can be imposed on a sample. This is possible (in mean) by controlling the displacement field through Kinematic Uniform Boundary Conditions (KUBC). These boundary conditions are classical in computational homogenization [22, 27].

Let consider Ω a regular open subset of $\mathbb{R}^2$ with smooth boundary denoted by $\partial\Omega$. Let denote by $\underline{\mathrm{u}}(\underline{\mathrm{x}})$ the displacement field in $\overline{\Omega}$, and $\underset{\sim}{\varepsilon}(\underline{\mathrm{x}})$ the related local strain state. The KUBC amounts to impose the following field at the boundary of the sample

$$\underline{\mathrm{u}}(\underline{\mathrm{x}}) = \underset{\sim}{\mathrm{E}} \cdot \underline{\mathrm{x}}, \quad \text{for } \underline{\mathrm{x}} \in \partial\Omega$$

in which $\underset{\sim}{\mathrm{E}}$ is a constant symmetric second order tensor. This implies that the mean strain over Ω is

$$\underset{\sim}{\mathrm{E}} = \left\langle \underset{\sim}{\varepsilon}(\underline{\mathrm{x}}) \right\rangle \quad \text{with} \quad \langle . \rangle = \frac{1}{V}\int_V \cdot \,\mathrm{d}V$$

The overall stress tensor $\underset{\sim}{\Sigma}$ is defined by the spatial average:

$$\underset{\sim}{\Sigma} = \left\langle \underset{\sim}{\sigma}(\underline{\mathrm{x}}) \right\rangle$$

It is worth noting that this type of boundary condition satisfies the Hill-Mandel Lemma. Hence, we have the following relation

$$\left\langle \underset{\sim}{\varepsilon} : \underset{\sim}{\sigma} \right\rangle = \underset{\sim}{\mathrm{E}} : \underset{\sim}{\Sigma}$$

If the local elasticity constitutive law is written

$$\underset{\sim}{\sigma} = \underset{\approx}{\mathrm{c}}(\underline{\mathrm{x}}) : \underset{\sim}{\varepsilon}$$

through this approach the mean quantities are related by an *effective* constant elasticity tensor $\underset{\approx}{\mathrm{C}}$:

$$\underset{\sim}{\Sigma} = \underset{\approx}{\mathrm{C}} : \underset{\sim}{\mathrm{E}}$$

Since, in our case, the material is supposed to be homogeneous, $\underset{\approx}{\mathrm{c}}$ is a constant tensor and

$$\underset{\approx}{\mathrm{C}} = \underset{\approx}{\mathrm{c}}$$

hence:

$$\underset{\sim}{\Sigma} = \underset{\approx}{\mathrm{c}} : \underset{\sim}{\mathrm{E}}$$

Hence the numerical experiments combined with the use of KUBC allow to directly measure tensor invariants. The dual boundary conditions also known as Static Uniform Boundary Conditions (SUBC) can also be used to control the problem in force rather than in displacement to identify the compliance tensor $\underset{\approx}{\mathrm{s}} = \underset{\approx}{\mathrm{c}}^{-1}$ [15][16].

Remark 5.1. ***The mechanical tests and measurements needed to identify the invariants are, in practice, far from being trivial. The development of testing devices that can impose enriched boundary condition would be of valuable interest. This development, if possible, would constitute a research direction in experimental mechanics on its own.***

[15] It is important to note that even if $\underset{\approx}{\mathrm{c}}$ and $\underset{\approx}{\mathrm{s}}$ admits harmonic decomposition, the relation between their different element are not direct at all.

[16] **In the present situation there is no need to use Periodic Boundary Conditions since the material we are testing is supposed to be homogeneous. Applying the procedure to an inhomogeneous periodic material will amount to construct invariant of the overall elasticity tensor, in such as case the use of PBC ensure the convergence of the effective properties making computations on an unique elementary cell.**

Applying this method, the 6 elementary invariants are obtained almost directly. As discussed previously such a set provides a unique *label* to each material. From this knowledge, the next step is to reconstruct the associate elasticity tensor in an appropriate basis.

6. Tensor reconstruction

The reconstruction is a two step process:

1. Construction of a tensor normal form;
2. Determination of the material orientation.

What we call a *tensor normal form* here is the expression of the tensor in a basis in which some properties are verified. In most of the cases the property is to have a maximal number of zero components. For material having $[\mathrm{D}_k]$-type symmetry classes associated bases coincide with the symmetry elements of the micro-structure. For $[\mathrm{Z}_k]$-type symmetry classes, the lack of mirror lines makes the choice of a normal form a bit more arbitrary but, nevertheless, a choice is still possible. It is worth noting that normal forms are not unique, hence some choices have to be made. Since these normal forms are related to specific bases, it is important to identify their angular positions with respect to the *testing device.* This is the second point of the reconstruction process.

6.1. Normal forms

In this section the fourth-order elasticity tensor $\underset{\approx}{\mathrm{C}}$ in $\mathbb{R}^2$ will be represented as a second order one $\underset{\sim}{\mathrm{C}}$ in $\mathbb{R}^3$. Details this construction is provided in Appendix B.

Symmetry class $[\mathrm{Z}_2]$

Let consider a generic elasticity tensor expressed in components in a randomly oriented basis:

$$[\underset{\sim}{\mathrm{C}}] = \begin{pmatrix} c_{1111} & c_{1122} & \sqrt{2}c_{1112} \\ & c_{2222} & \sqrt{2}c_{2212} \\ & & 2c_{1212} \end{pmatrix}$$

this tensor can also be expressed (in the same basis) in terms of its harmonic components

$$[\underset{\sim}{\mathrm{C}}] = \begin{pmatrix} K+G+a_1+d_1 & K-G-d_1 & \sqrt{2}\frac{a_2+2d_2}{2} \\ & K+G-a_1+d_1 & \sqrt{2}\frac{a_2-2d_2}{2} \\ & & 2G-2d_1 \end{pmatrix}$$

For the generic class, the normal form will be chosen such as $a_2 = 0$ and $a_1 > 0$. The reason of this choice, it that the associated formula for the reconstruction are simpler. Hence, the normal form has the following shape

$$\underset{\sim}{[\mathrm{C}]}^{\mathrm{Z}_2} = \begin{pmatrix} K+G+a_1+d_1 & K-G-d_1 & \sqrt{2}d_2 \\ & K+G-a_1+d_1 & -\sqrt{2}d_2 \\ & & 2G-2d_1 \end{pmatrix}$$

To proceed the reconstruction we need to evaluate the invariants for this normal form.

$$\begin{cases} I_2 = 2a_1^2 \\ J_2 = 8(d_1^2 + d_2^2) \\ I_3 = 4a_1^2 d_1 \\ J_3 = -4a_1^2 d_2 \end{cases}$$

Inverting the system, and according to our choice selecting the positive root for a_1:

$$\begin{cases} a_1 = \frac{\sqrt{2I_2}}{2} \\ d_1 = \frac{I_3}{2I_2} \\ d_2 = -\frac{J_3}{2I_2} \end{cases}$$

Hence

$$\underset{\sim}{[\mathrm{C}]}^{\mathrm{Z}_2} = \begin{pmatrix} I_1+\frac{J_1}{2}+\frac{\sqrt{2I_2}}{2}+\frac{I_3}{2I_2} & I_1-\frac{J_1}{2}-\frac{I_3}{2I_2} & -\frac{\sqrt{2}J_3}{2I_2} \\ & I_1+\frac{J_1}{2}-\frac{\sqrt{2I_2}}{2}+\frac{I_3}{2I_2} & \frac{\sqrt{2}J_3}{2I_2} \\ & & J_1-\frac{I_3}{I_2} \end{pmatrix}$$

It can be observed that we did not exploit the J_2 invariant. An interesting and a bit tricky situation appears here. In fact using J_2 leads to the following expression for d_2:

$$d_2 = \frac{1}{2\sqrt{2}}\sqrt{\frac{J_2 I_2^2 - 2I_3^2}{I_2^2}}$$

According to the syzygy

$$J_2 I_2^2 - 2I_3^2 - 2J_3^2 = 0$$

this relation can be recast

$$d_2 = \frac{\sqrt{J_3^2}}{2I_2}$$

hence two possibilities appear concerning the choice of the root for J_3^2. If we consider SO(2)-action (only rotating the sample), the two different roots are two different SO(2)-orbits. Hence the sign of J_3 distinguish between these two orbits. But if we consider O(2)-action we can also *flip* the sample, and as a consequence the two roots of J_3 belong to the same O(2)-orbit. So to label O(2)-orbit the following two sets can be used: (I_2, J_2, I_3), (I_2, I_3, J_3), while for SO(2)-orbit this choice reduces only to (I_2, I_3, J_3).

Symmetry class $[\mathrm{D}_2]$

For this class, the choice of a normal form will be chosen so as $a_2 = d_2 = 0$. In this case the *vertical* components of $\underset{\sim}{\mathrm{a}}$ and $\underset{\sim}{\mathrm{D}}$ are null. According to their sign, this conducts to 4 different situations for the couple (a_1, d_1):

$$(+,+) \quad , \quad (+,-), \quad (-,-), \quad (-,+)$$

But in fact, as depicted on fig.3 and fig.4, these 4 situations correspond to 2 different orbits according to the sign of d_1 which is determined by I_3. Hence the supplementary condition that $a_1 > 0$ will be added to retain in each case on unique normal form (the other one being deduced by a material rotation of $\frac{\pi}{2}$).

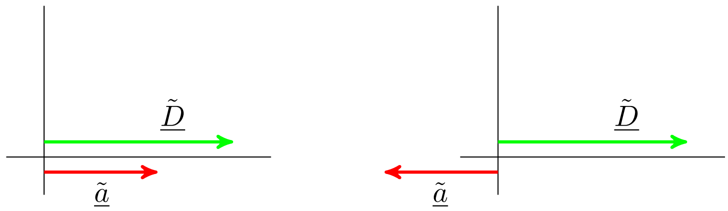


Figure 3: Two configurations on the same orbit, $I_3 > 0$. Those configurations are related by a **material** rotation of $\frac{\pi}{2}$

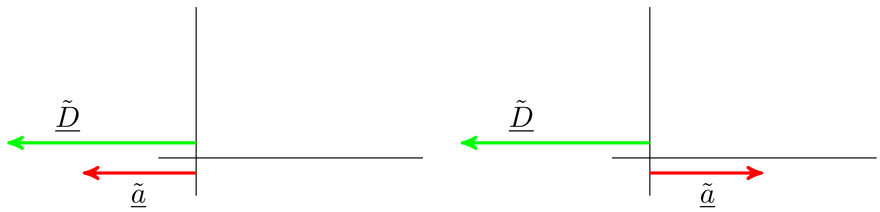


Figure 4: Two configurations on the same orbit, $I_3 < 0$. Those configurations are related by a **material** rotation of $\frac{\pi}{2}$

To proceed the reconstruction we need to evaluate the invariants for this normal form.

$$\begin{cases} I_2 = 2a_1^2 \\ J_2 = 8d_1^2 \\ I_3 = 4a_1^2 d_1 \end{cases} \Rightarrow \begin{cases} d_1 = \frac{I_3}{2I_2} \\ a_1 = \frac{\sqrt{2I_2}}{2} \end{cases}$$

As a consequence

$$[\underset{\sim}{\mathrm{C}}]^{\mathrm{D}_2} = \begin{pmatrix} K+G+a_1+d_1 & K-G-d_1 & 0 \\ & K+G-a_1+d_1 & 0 \\ & & 2G-2d_1 \end{pmatrix} = \begin{pmatrix} I_1+\frac{J_1}{2}+\frac{\sqrt{2I_2}}{2}+\frac{I_3}{2I_2} & I_1-\frac{J_1}{2}-\frac{I_3}{2I_2} & 0 \\ & I_1+\frac{J_1}{2}-\frac{\sqrt{2I_2}}{2}+\frac{I_3}{2I_2} & 0 \\ & & J_1-\frac{I_3}{I_2} \end{pmatrix}$$

It can be observed that the set (I_2, J_2) is unable to label an orthotropic material uniquely since it can not make distinction between the two orbits represented on fig.3 and fig.4, while this distinction can be made with the set (I_2, I_3).

Symmetry class $[\mathrm{D}_4]$

In the $[\mathrm{D}_4]$ situation, we choose for the normal form $d_2 = 0, d_1 > 0$. We have:

$$[\underset{\sim}{\mathrm{C}}]^{\mathrm{D}_4} = \begin{pmatrix} K+G+d_1 & K-G-d_1 & 0 \\ & K+G+d_1 & 0 \\ & & 2G-2d_1 \end{pmatrix}$$

hence the evaluation of the invariants on this slice gives

$$\begin{cases} I_2 = 0 \\ J_2 = 8d_1^2 \\ I_3 = 0 \end{cases}$$

And choosing the positive square root of J_2:

$$[\underset{\sim}{\mathrm{C}}]^{\mathrm{D}_4} = \begin{pmatrix} I_1+\frac{J_1}{2}+\frac{\sqrt{8J_2}}{8} & I_1-\frac{J_1}{2}-\frac{\sqrt{8J_2}}{8} & 0 \\ & I_1+\frac{J_1}{2}+\frac{\sqrt{8J_2}}{8} & 0 \\ & & J_1-\frac{\sqrt{8J_2}}{4} \end{pmatrix}$$

It can be observed that the choice of the negative square root would have defined another normal form on the same orbit obtained by a rotation of $\frac{\pi}{4}$.

Symmetry class $[\mathrm{O}(2)]$

The last situation is trivial

$$[\underset{\sim}{\mathrm{C}}]^{\mathrm{O}(2)} = \begin{pmatrix} K+G & K-G & 0 \\ & K+G & 0 \\ & & G \end{pmatrix} = \begin{pmatrix} I_1+\frac{J_1}{2} & I_1-\frac{J_1}{2} & 0 \\ & I_1+\frac{J_1}{2} & 0 \\ & & J_1 \end{pmatrix}$$

6.2. Determination of the rotation

For the $[\mathrm{Z}_2]$ and $[\mathrm{D}_2]$ symmetry classes, bases for normal forms are defined by the condition $a_2 = 0$. Let consider our material in the testing basis, which is different from the normal one. The tensor $\underset{\sim}{\mathrm{a}}$ can be constructed in the following way

$$\underset{\sim}{\mathrm{a}} = \underset{\sim}{\sigma}^d\,(0;1,1,0) = \begin{pmatrix} a_1 & a_2 \\ a_2 & -a_1 \end{pmatrix}$$

Hence

$$\cos\alpha = \frac{a_1}{\sqrt{a_1^2 + a_2^2}} \quad ; \quad \sin\alpha = \frac{a_2}{\sqrt{a_1^2 + a_2^2}}$$

and so

$$\cos\alpha = \frac{\sqrt{2}a_1}{\sqrt{I_2}} \quad ; \quad \sin\alpha = \frac{\sqrt{2}a_2}{\sqrt{I_2}}$$

Since $\underset{\sim}{\mathrm{a}} \in \mathbb{K}^2$ when the elasticity tensor is rotated by θ $\underset{\sim}{\mathrm{a}}$ is rotated by 2θ. Hence the basis of the normal form is oriented with an angle $\frac{\alpha}{2}$ with respect to the testing device.

For the $[\mathrm{D}_4]$ symmetry class, the covariant $\underset{\sim}{\mathrm{a}}$ is null and the basis forthe normal form is defined by the condition $d_2 = 0$. In this specific situation information about $\underset{\approx}{\mathrm{D}}$ can be obtained in the following way

$$\underset{\sim}{\sigma}^d\left(0;0,0,\frac{1}{2}\right) = \begin{pmatrix} d_2 & G - d_1 \\ G - d_1 & -d_2 \end{pmatrix}$$

Hence

$$\cos\beta = \frac{d_1}{\sqrt{d_1^2 + d_2^2}} \quad ; \quad \sin\beta = \frac{d_2}{\sqrt{d_1^2 + d_2^2}}$$

and so

$$\cos\beta = \frac{\sqrt{8}d_1}{\sqrt{J_2}} \quad ; \quad \sin\beta = \frac{\sqrt{8}d_2}{\sqrt{J_2}}$$

Since $\underset{\approx}{\mathrm{D}} \in \mathbb{K}^4$ when the elasticity tensor is rotated by θ, $\underset{\approx}{\mathrm{D}}$ is rotated by 4θ. Hence the basis of the normal form is oriented with an angle $\frac{\beta}{4}$ with respect to the testing device.

7. Conclusion

In the present paper an identification procedure of the elastic material parameters has been proposed. The method is based on the (almost direct) evaluation of the invariants of the 2D elasticity tensor rather than on its tensor components. Such a way to proceed is appealing since all the quantities that determine the elastic material are obtained in the same time: symmetry class, material orientation, material parameters. Further works will be devoted to compare the procedure proposed in this paper to more classical techniques through numerical studies.

One natural question is how to extend this approach to 3D elasticity. This problem is far more complicated since the minimal integrity basis is constituted of 299 elements [28], and the size of a functional basis is presently unknown. Furthermore the elementary covariants are no more characterized, as in 2D, by only one invariant. For example, in 3D, $\underset{\approx}{\mathrm{D}} \in \mathbb{H}^4$ is characterized by 9 invariants [10]. Hence the extension of the present method to 3D is not direct. An intermediate situation of interest is the study of O(2)-invariants polynomials of the 3D elasticity tensor. In such as case a functional basis is known [16]. Another interesting extension of the present work would be to consider 2D generalized continuum model [2, 3].

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Appendix A. Harmonic decomposition

In order to properly set the problem, the tensor spaces under study should be decomposed into a collection of elementary spaces. The nature of those elementary spaces depends on the considered group action. In the present case, our fundamental pieces are O(2)-irreducible spaces. In the mechanical literature this decomposition is often referred to as the harmonic decomposition [6, 25, 14, 17].

Appendix A.1. One basic example

Let consider the case of a second-order symmetric tensor, as well known any $\mathrm{T}_{(ij)} \in \mathbb{T}_{(ij)}$ admits the following decomposition

$$T_{ij} = K^2_{ij} + \frac{1}{2}K^0\delta_{ij} = \phi(K^2_{ij}, K^0)$$

where $\mathrm{K}^2 \in \mathbb{K}^2$ and $\mathrm{K}^0 \in \mathbb{K}^0$ are, respectively, the 2-D deviatoric and 1-D spheric part of $\mathrm{T}_{(ij)}$. They are defined by the following formula:

$$K^0 = T_{ii} \quad ; \quad K^2_{ij} = K_{ij} - \frac{1}{2}K^0\delta_{ij}$$

ϕ, defined by the expression (Appendix A.1), is an isomorphism between $\mathbb{T}_{(ij)}$ and the direct sum of $\mathbb{K}^2$ and $\mathbb{K}^0$

$$\mathbb{T}_{(ij)} \simeq \mathbb{K}^2 \oplus \mathbb{K}^0$$

The main property of this decomposition is to be O(2)-invariant, or expressed in other way the components $(\mathrm{K}^0, \underset{\sim}{\mathrm{K}}^2)$ are covariant with $\underset{\sim}{\mathrm{T}}$ under O(2)-action, i.e.

$$\forall \underset{\sim}{\mathrm{Q}} \in \mathrm{O}(2),\ \forall \underset{\sim}{\mathrm{T}} \in \mathbb{T}_{(ij)}, \qquad \underset{\sim}{\mathrm{Q}}\underset{\sim}{\mathrm{T}}\underset{\sim}{\mathrm{Q}}^T = \phi(\underset{\sim}{\mathrm{Q}}\underset{\sim}{\mathrm{K}}^2\underset{\sim}{\mathrm{Q}}^T, \mathrm{K}^0)$$

This decomposition is irreducible meaning that those tensors can not be split into smaller ones satisfying again this property. Irreducible tensors that satisfy this property are said to be harmonic.

Appendix A.2. The harmonic decomposition

The harmonic decomposition establishes an isomorphism between $\mathbb{T}^{(n)}$ and a direct sum of harmonic tensor spaces $\mathbb{K}^k$ [14]. We shall note

$$\mathbb{T}^{(n)} \simeq \bigoplus_{k=-1}^{n} \left(\bigoplus_{l=0}^{\alpha_k} \right) \mathbb{K}^k \tag{A.1}$$

where k denotes the order of the harmonic space and α_k indicates the multiplicity of $\mathbb{K}^k$ in the decomposition. To spare space, this decomposition will often be written

$$\mathbb{T}^{(n)} \simeq \bigoplus_{k=-1}^{n} \alpha_k \mathbb{K}^k \tag{A.2}$$

As a consequence, any element of $\mathbb{T}^{(n)}$ can be expressed as

$$\mathrm{T}^{(n)} = \sum_{k=0}^{n} \left(\sum_{l=0}^{\alpha_k} D_{k,l}(n) \right) \tag{A.3}$$

in which tensor $D_{k,l}(n)$ is the l-th elements of order k imbedded into a n-th order tensor. It is worth noting that the explicit decomposition (A.3) is uniquely defined only if $\alpha_k \leq 1$ [19]. At the opposite the global structure of the decomposition (A.2) is uniquely defined.

Appendix B. Matrix representations of the elasticity tensor

Let be defined the following spaces:

$$\mathbb{T}_{(ij)} = \{ \mathrm{T} \in \mathbb{T}_{ij} | \mathrm{T} = \sum_{i,j=1}^{2} T_{ij} \mathbf{e}_i \otimes \mathbf{e}_j,\ T_{ij} = T_{ji} \}$$

which is, in 2D, respectively, a 3D vector spaces. Therefore the elasticity tensor $\underset{\approx}{\mathrm{C}}$ is a self-adjoint endomorphism of $\mathbb{T}_{(ij)}$.

In order to express the Cauchy-stress tensor $\underset{\sim}{\sigma}$, the strain tensor ε as 3-dimensional vectors and write $\underset{\approx}{\mathrm{C}}$ as a 3×3 we introduce the following orthonormal basis vectors:

$$\widetilde{\mathbf{e}}_I = \left(\frac{1-\delta_{ij}}{\sqrt{2}} + \frac{\delta_{ij}}{2} \right) (\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i), \quad 1 \leq I \leq 3$$

where the summation convention for a repeated subscript does not apply. Then, the aforementioned tensors can be expressed as:

$$\widetilde{\varepsilon} = \sum_{I=1}^{3} \widetilde{\varepsilon}_I \widetilde{\mathbf{e}}_I, \quad \widetilde{\sigma} = \sum_{I=1}^{3} \widetilde{\sigma}_I \widetilde{\mathbf{e}}_I, \quad \widetilde{\mathbf{C}} = \sum_{I,J=1,1}^{3,3} \widetilde{C}_{IJ} \widetilde{\mathbf{e}}_I \otimes \widetilde{e}_J \tag{B.1}$$

so that the elastic relation can be written in the matrix form

$$\widetilde{\sigma}_I = \widetilde{C}_{IJ} \widetilde{\varepsilon}_J$$

The relationship between the matrix components $\widetilde{\varepsilon}_I$ and ε_{ij} is

$$\widetilde{\varepsilon}_I = \begin{cases} \varepsilon_{ij} & \text{if } \ i = j, \\ \sqrt{2}\varepsilon_{ij} & \text{if } \ i \neq j; \end{cases} \tag{B.2}$$

and, obviously, the same relation between $\widetilde{\sigma}_I$ and σ_{ij} hold. For the constitutive tensor we have the following correspondence:

$$\widetilde{C}_{IJ} = \begin{cases} C_{ijkl} \ \text{ if } \ i = j \text{ and } k = l, \\ \sqrt{2}C_{ijkl} \ \text{ if } \ i \neq j \text{ and } k = l \text{ or } i = j \text{ and } k \neq l, \\ 2C_{ijkl} \ \text{ if } \ i \neq j \text{ and } k \neq l. \end{cases} \tag{B.3}$$

It remains to choose an appropriate two-to-one subscript correspondences between ij and I:

I	1	2	3
ij	11	22	12

Table B.1: The two-to-one subscript correspondence for 2D strain/stress tensors

Hence we obtain the following second-order tensor representation for $\underset{\approx}{\mathrm{C}}$

$$\widetilde{\mathrm{C}} = \begin{pmatrix} c_{1111} & c_{1122} & \sqrt{2}c_{1112} \\ c_{1122} & c_{2222} & \sqrt{2}c_{2212} \\ \sqrt{2}c_{1112} & \sqrt{2}c_{2212} & 2c_{1212} \end{pmatrix}$$

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